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Some Existence Results for Vector Quasivariational Inequalities Involving Multifunctions and Applications to Traffic Equilibrium Problems★

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Abstract. Some existence results for vector quasivariational inequalities with multifunctions in Banach spaces are derived by employing the KKM-Fan theorem. In particular, we generalize a result by Lin, Yang and Yao, and avoid monotonicity assumptions. We also consider a new quasivariational inequality problem and propose notions of weak and strong equilibria while applying the results to traffic network problems.

Key words: Generalized upper or lower hemicontinuity, Multifunctions, Pseudomonotonicity, Traffic networks, Upper semicontinuity, Vector quasivariational inequalities, Weak and strong equilibria

1. Introduction and preliminaries

Vector variational inequalities were proposed by Giannessi (1980) and have been intensively developed in the last two decades.

The existence of a solution of vector variational inequalities is probably most intensively studied. Among numerous techniques used in deriving existence results, the well-known KKM-Fan theorem is one of the most helpful tools (see e.g., Hadjisavvas and Schaible, 1996, 1998; Ding (1997); Lin et al., 1997; Konnov, 1998; Ansari, 2000; John, 2001; Diafari-Rouhani et al., 2001; Fu and Wan, 2002; Khanh and Luu, 2004; Hai and Khanh, in press). Besides, assumptions about some monotonicity properties are often inevitably imposed. The usual monotonicity was first used (see e.g., Blum and Oetti, 1993). Then many authors assumed pseudomonotonicity (e.g., Yao, 1994; Lin et al., 1997; Diafari-Rouhani et al., 2001; Khanh and Luu, 2004; Hai and Khanh, in press). More relaxed

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assumptions of quasimonotonicity type were proposed in Hadjisavvas and Schaible (1996, 1998), John (2001) and Konnov, (1998).

In the present note we use the KKM-Fan theorem and an equivalent fixed point theorem to get existence results for rather general vector quasivariational inequalities involving multifunctions. For a commonly studied quasivariational inequality problem, we try to weaken assumptions to get rid of (relaxed) monotonicity. We consider also a new type of quasivariational inequalities. Extending the Wardrop equilibrium, we propose definitions of weak equilibrium and strong equilibrium for traffic networks with multivalued costs and apply our existence results to obtain the existence of such equilibria.

For the sake of comparison let us stop in several lines to mention several other kinds of results and techniques. Ricceri (1995) supplied a survey of existence results based on the closedness of certain level sets. Guo and Yao (1994) used the so-called class $(S)_+$ of mappings (introduced by Browder (1970), which are demicontinuous or continuous on finite-dimensional subspaces instead of the class of pseudomonotone and upper semicontinuous (usc) mappings.

Yao (1994) assumed the mappings involved in the variational inequality problem to be pseudomonotone and continuous on finite-dimentional subspaces and reduced the problem to the finite-dimensional case so that the classical Stampacchia theorem can be applied. Crouzeix (1997) used a gap function technique to prove that the solution set is nonempty and compact (in R^n) under the assumption that the mapping of variational inequalities is monotone, usc and has convex compact values. Zhao et al. (1999) considered variational inequalities with the constraint sets defined by convex inequalities and affine equalities. They developed a concept of exceptional family (a variational inequality problem does not have exceptional families means that some kind of coercivity is satisfied) and used the Karush–Kuhn–Tucker theorem to prove that the nonexistence of an exceptional family is a sufficient condition for the existence of a solution if the mapping is continuous.

The problem under our consideration is as follows. Let X and Y be real Banach spaces, $A \subset X$ be a nonempty, closed and convex subset. Let $C: A \overrightarrow{\rightarrow} Y$ be a multifunction with values being closed and convex cones, different from Y and with nonempty interiors. Let $K: A \overrightarrow{\rightarrow} X$ be a multifunction with nonempty convex values. Let $T: A \overrightarrow{\rightarrow} L(X, Y)$, where $L(X, Y)$ stands for the space of all continuous linear mappings from X to Y. Let $f: A \times A \rightarrow Y$ be a mapping with $f(x, x) \in C(x) \cap -C(x)$, $\forall x \in A$. Assume that $Y \backslash -\text{int}\, C(.)$ is a weakly closed mapping, i.e., its graph is closed in $X \times Y$ with the weak topologies of X and Y. We consider the following two vector quasivariational inequality problems of:

(QVI) : Finding $\bar{x} \in A \bigcap cl\, K(\bar{x})$ such that, for each $x \in K(\bar{x})$, there is $\bar{t} \in T(\bar{x})$ such that

$$(\bar{t}, x - \bar{x}) + f(x, \bar{x}) \in Y \backslash -\text{int}\, C(\bar{x});$$

(SQVI): Finding $\bar{x} \in A \bigcap cl\, K(\bar{x})$ such that, for each $x \in K(\bar{x})$ and for each $t \in T(\bar{x})$,

$$(t, x - \bar{x}) + f(x, \bar{x}) \in Y \backslash -\text{int}\, C(\bar{x}).$$

Here cl means the closure and (t, x) means the value of linear mapping $t \in L(X, Y)$ at $x \in X$. Note that (QVI) was considered by many authors, while (SQVI) has not been studied.

We recall first some definitions needed in the sequel. A multifunction $F: X \overrightarrow{\rightarrow} Y$ is said to be usc at $x_0 \in \text{dom} F := \{x \in X : F(x) \neq \emptyset\}$ if, for each neighborhood U of $F(x_0)$, there is a neighborhood N of x_0 such that $F(N) \subset U$. F is called usc in $A \subset X$ if it is usc at every $x \in A$ and is called usc if it is usc at every $x \in \text{dom} F$. All other definitions for a point will be extended to a set by this way. F is said to be lower semicontinuous (lsc) at $x_0 \in \text{dom} F$ if, for each open subset U satisfying $U \cap F(x_0) \neq \emptyset$, there exists a neighborhood N of x_0 such that $U \cap F(x) \neq \emptyset$ for all $x \in N$. F is said to be upper hemicontinuous, uhc for short, (lower hemicontinuous lhc for short), at x_0 if, for each $x \in X$, the multifunction $\alpha \mapsto F(\alpha x + (1 - \alpha) x_0)$ is usc (lsc, respectively) at 0^+. A multifunction $T: A \overrightarrow{\rightarrow} L(X, Y)$ is called generalized upper hemicontinuous, guhc for short, (generalized lower hemicontinuous, glhc for short), at $x_0 \in A$ if, for each $x \in A, \alpha \mapsto (T(\alpha x + (1 - \alpha) x_0), x - x_0)$ is usc (lsc, respectively) at 0^+.

Assume that C is a convex cone of Y. A mapping $f: X \to Y$ is called C-convex in a convex subset $A \subset X$ if, $\forall x_1 \in A, \forall x_2 \in A, \forall \gamma \in [0, 1]$,

$$(1 - \gamma) f(x_1) + \gamma f(x_2) - f((1 - \gamma) x_1 + \gamma x_2) \in C.$$

We need also the following notion of pseudomonotonicity. A pair (T, f) of $T: A \overrightarrow{\rightarrow} L(X, Y)$ and $f: A \times A \to Y$ is said to be pseudomonotone in A if, $\forall x \in A, \forall y \in A$,

$$\begin{aligned} &[\exists s \in T(x), (s, y - x) + f(y, x) \in Y \backslash -\text{int}\, C(x)] \\ \Rightarrow &[\forall t \in T(y), (t, y - x) + f(y, x) \in Y \backslash -\text{int}\, C(x)]. \end{aligned}$$

(T, f) is called weakly pseudomonotone if "$\forall t$" in the above statement is replaced by "$\exists t$".

A multifunction H of a subset A of a topological vector space X into X is termed a KKM mapping in A if, for each finite subset $\{x_1, \ldots, x_n\}$ of A, one has $co\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} H(x_i)$, where $co\{\cdot\}$ stands for the convex hull.

The main tool for proving our results is the following well-known KKM-Fan theorem in Fan (1984).

THEOREM 1.1. *Assume that X is a topological vector space, $A \subset X$ is nonempty and $H: A \overrightarrow{\to} X$ is a KKM mapping with closed values. If there is a subset X_0 contained in a compact convex subset of A such that $\cap_{x \in x_0} H(x)$ is compact, then $\cap_{x \in A} H(x) \neq \emptyset$.*

Theorem 1.1 has many equivalent formulations in terms of fixed points. The following result is a slightly weaker version (suitable for our use) of Tarafdar's theorem in Tarafdar (1987), which is equivalent to Theorem 1.1.

THEOREM 1.2. *Let X be a Hausdorff topological vector space and $A \subset X$ be nonempty. Let $\phi: A \overrightarrow{\to} A$ have nonempty convex values. Assume that*

(i) *$\phi^{-1}(y)$ is open in A for each $y \in A$;*
(ii) *there exists a nonempty subset X_0 contained in a compact convex subset of A such that $A \setminus \cup_{y \in X_0} \phi^{-1}(y)$ is compact or empty.*

Then, there is a fixed point x_0 of ϕ in A, i.e., $x_0 \in \phi(x_0)$.

2. Existence for (QVI)

In the sequel let $E := \{x \in A : x \in cl K(x)\}$.

The first result below is an extention of the main result in Lin et al. (1997).

THEOREM 2.1. *Assume, for problem (QVI),*

(i) *T is guhc in A and has nonempty compact values;*
(ii) *(T, f) is weakly pseudomonotone in A and, $\forall x \in A$, $f(., x)$ is $C(x)$-convex in A;*
(iii) *$\forall x \in A, \forall y \in A, \forall x_\alpha \stackrel{w}{\to} x, \exists x_\beta$ (subnet), $\exists u \in -C(x) + f(y, x)$, $f(y, x_\beta) \stackrel{w}{\to} u$, where $\stackrel{w}{\to}$ means the weak convergence;*
(iv) *for all $x \in A$, $A \cap K(x)$ is nonempty, $K^{-1}(x)$ is weakly open in A, $cl K(.)$ is weakly closed; moreover, $\forall x \in E, \forall y \in K(x), \forall \gamma \in (0, 1], \gamma y + (1-\gamma)x \in K(x)$;*
(v) *there is a nonempty weakly compact subset D of A and a subset X_0 of a weakly compact convex subset of A such that $\forall x \in A \setminus D, \exists z \in X_0 \cap K(x), (T(z), z - x) + f(z, x) \subset -\text{int}\, C(x)$.*

Then, (QVI) has solutions.

Proof. For $x, y \in A$ and $i = 1, 2,$ set

$$P_1(x) := \{z \in A : (T(x), z - x) + f(z, x) \subset -\text{int}\, C(x)\},$$

$$P_2(x) := \{z \in A : (T(z), z - x) + f(z, x) \subset -\mathrm{int}\, C(x)\},$$

$$\Phi_i(x) := \begin{cases} K(x) \bigcap P_i(x) & \text{if } x \in E, \\ A \bigcap K(x) & \text{if } x \in A \backslash E, \end{cases}$$

$$Q_i(y) := A \setminus \Phi_i^{-1}(y).$$

Since $A \bigcap K(x) \neq \emptyset$ for all $x \in A, \bigcup_{y \in A} K^{-1}(y) = A$. Theorem 1.2 in turn guarantees that $K(.)$ has a fixed point in A (and hence $E \neq \emptyset$). Indeed, only assumption (ii) of this theorem is to be checked. By (v),

$$A \backslash D \subset \bigcup_{x \in X_0} K^{-1}(x) \subset A.$$

Hence, $A \setminus \bigcup_{x \in X_0} K^{-1}(x)$ is contained in D and is then weakly compact. Then, since $cl K(.)$ is weakly closed, so is E.

Next we calculate $Q_i(y)$, for $i = 1, 2$ and for $y \in A$. We have

$$\begin{aligned} \Phi_i^{-1}(y) &= \{x \in A : y \in \Phi_i(x)\} \\ &= \{x \in E : y \in K(x) \bigcap P_i(x)\} \bigcup \{x \in A \backslash E : y \in K(x)\} \\ &= \Big[E \bigcap K^{-1}(y) \bigcap P_i^{-1}(y)\Big] \bigcup \Big[(A \backslash E) \bigcap K^{-1}(y)\Big] \\ &= \Big[\Big(E \bigcap P_i^{-1}(y)\Big) \bigcup (A \backslash E)\Big] \bigcap K^{-1}(y) \\ &= \Big[(A \backslash E) \bigcup P_i^{-1}(y)\Big] \bigcap K^{-1}(y). \end{aligned}$$

Therefore,

$$\begin{aligned} Q_i(y) &= \Big\{A \setminus \Big[(A \backslash E) \bigcup P_i^{-1}(y)\Big]\Big\} \bigcup \big[A \backslash K^{-1}(y)\big] \\ &= \Big[E \bigcap \Big(A \backslash P_i^{-1}(y)\Big)\Big] \bigcup \big[A \backslash K^{-1}(y)\big]. \end{aligned} \tag{1}$$

Now we show that $Q_1(.)$ is a KKM mapping in A. Suppose to the contrary that there is a convex combination in $A, \hat{x} = \sum_{j=1}^{n} \alpha_j y_j$, such that $\hat{x} \notin \bigcup_{j=1}^{n} Q_1(y_j)$, i.e., $\hat{x} \in \Phi_1^{-1}(y_j)$ and then $y_j \in \Phi_1(\hat{x})$ for all $j = 1, \ldots, n$. If $\hat{x} \in E$, then $\Phi_1(\hat{x}) = K(\hat{x}) \bigcap P_1(\hat{x})$. Hence, $y_j \in P_1(\hat{x})$, i.e.,

$$(T(\hat{x}), y_j - \hat{x}) + f(y_j, \hat{x}) \subset -\mathrm{int}\, C(\hat{x}).$$

Therefore, $\forall t \in T(\hat{x})$,

$$0 = (t, \hat{x} - \hat{x}) = \sum_{j=1}^{n} \alpha_j [(t, y_j - \hat{x}) + f(y_j, \hat{x})] - \sum_{j=1}^{n} \alpha_j f(y_j, \hat{x})$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \alpha_j [(t, y_j - \hat{x}) + f(y_j, \hat{x})] + \left[f\left(\sum_{j=1}^{n} \alpha_j y_j, \hat{x} \right) - \sum_{j=1}^{n} \alpha_j f(y_j, \hat{x}) \right] \\
&\quad - f(\hat{x}, \hat{x}) \in -\mathrm{int}\, C(\hat{x}) - C(\hat{x}) - C(\hat{x}) = -\mathrm{int}\, C(\hat{x}).
\end{aligned}
$$

Consequently, $C(\hat{x}) = Y$, a contradiction. For the remaining case $\hat{x} \in A \backslash E$, i.e., $\hat{x} \notin cl K(\hat{x})$, one sees that, by the definition of Φ_1, $y_j \in \Phi_1(\hat{x}) = A \bigcap K(\hat{x})$. Hence, $\hat{x} = \sum_{j=1}^{n} \alpha_j y_j \in K(\hat{x})$, a contradiction again. Thus, Q_1 is a KKM mapping.

On the other hand, by the definition of P_i and by the weak pseudomonotonicity of (T, f), $A \backslash P_1^{-1}(y) \subset A \backslash P_2^{-1}(y)$. Hence, $Q_1(y) \subset Q_2(y), \forall y \in A$. Thus, Q_2 is also a KKM mapping in A.

To apply Theorem 1.1 we check the weak closedness of $Q_2(y)$ for each $y \in A$. Using (1) it remains to check the weak closedness of $A \backslash P_2^{-1}(y), \forall y \in A$. Assume that $x_\alpha \xrightarrow{w} x \in A$ and $x_\alpha \in A \backslash P_2^{-1}(y)$, i.e., $\exists t_\alpha \in T(y)$ such that

$$
(t_\alpha, y - x_\alpha) + f(y, x_\alpha) \in Y \setminus -\mathrm{int}\, C(x_\alpha). \tag{2}
$$

By the compactness of $T(y)$, there are $t \in T(y)$ and a subnet $t_\beta \to t$. One has

$$
(t_\beta, y - x_\beta) = (t_\beta - t, y - x_\beta) + (t, y - x_\beta). \tag{3}
$$

Since t is also continuous, when X and Y are equipped with the weak topologies, $(t, y - x_\beta) \xrightarrow{w} (t, y - x)$. On the other hand,

$$
\|(t_\beta - t, y - x_\beta)\| \leqslant \|t_\beta - t\| \, \|y - x_\beta\| \to 0
$$

as $\|y - x_\beta\|$ is bounded. Therefore (3) implies that $(t_\beta, y - x_\beta) \xrightarrow{w} (t, y - x)$. Taking (iii) into account, there is a subnet x_γ of x_β and $u \in -C(x) + f(y, x)$ such that $f(y, x_\gamma) \xrightarrow{w} u$. Now the weak closedness of $Y \setminus -\mathrm{int}\, C(.)$ and (2) together imply that

$$
(t, y - x) + u \in Y \setminus -\mathrm{int}\, C(x).
$$

Hence,

$$
\begin{aligned}
(t, y - x) + f(y, x) &= (t, y - x) + u + f(y, x) - u \\
&\in Y \setminus -\mathrm{int}\, C(x) + C(x) = Y \setminus -\mathrm{int}\, C(x),
\end{aligned}
$$

which shows that $x \in A \setminus P_2^{-1}(y)$ and then $A \backslash P_2^{-1}(y)$ is weakly closed. Thus,

$Q_2(y)$ is weakly closed for all $y \in A$. Moreover, by (v), $\forall x \in A \setminus D, \exists z \in X_0 \bigcap K(x)$ such that $z \in \Phi_2(x)$. Consequently, $A \setminus D \subset \bigcup_{z \in X_0} \Phi_2^{-1}(z)$ and hence

$$D \supset \bigcap_{z \in X_0} A \setminus \Phi_2^{-1}(z) = \bigcap_{z \in X_0} Q_2(z).$$

With this, all the assumptions of Theorem 1.1 are satisfied and then there exists $\bar{x}$ of

$$\bigcap_{y \in A} (A \setminus \Phi_2^{-1}(y)) = A \setminus \bigcup_{y \in A} \Phi_2^{-1}(y). \tag{4}$$

Finally, we show that $\bar{x}$ is a solution of (QVI). By (4), $\Phi_2(\bar{x}) = \emptyset$. Two possibilities arise. If $\bar{x} \in A \setminus E$, by (iv), $\Phi_2(\bar{x}) = A \bigcap K(\bar{x}) \neq \emptyset$, a contradiction. Otherwise, i.e., $\bar{x} \in E, \emptyset = \Phi_2(\bar{x}) = K(\bar{x}) \bigcap P_2(\bar{x})$. So, for each $y \in K(\bar{x}), y \notin P_2(\bar{x})$, i.e., $\exists \bar{t} \in T(y)$,

$$(\bar{t}, y - \bar{x}) + f(y, \bar{x}) \subset Y \setminus -\text{int}\, C(\bar{x}). \tag{5}$$

Suppose $\bar{x}$ is not a solution, i.e., $\exists \bar{y} \in K(\bar{x}), \forall s \in T(\bar{x})$,

$$(s, \bar{y} - \bar{x}) + f(\bar{y}, \bar{x}) \in -\text{int}\, C(\bar{x}).$$

This and the guhc of T imply, for all $\lambda > 0$ small enough, that

$$(T(\lambda \bar{y} + (1-\lambda)\bar{x}), \bar{y} - \bar{x}) + f(\bar{y}, \bar{x}) \subset -\text{int}\, C(\bar{x}). \tag{6}$$

On the other hand, by (5), (iv) and the $C(\bar{x})$-convexity of $f(., \bar{x})$, one has

$$\begin{aligned}
&(\bar{t}, \bar{y} - \bar{x}) + f(\bar{y}, \bar{x}) \\
&= \frac{1}{\lambda}[(\bar{t}, \lambda \bar{y} + (1-\lambda)\bar{x} - \bar{x}) + f(\lambda \bar{y} + (1-\lambda)\bar{x}, \bar{x})] \\
&\quad + \frac{1}{\lambda}[\lambda f(\bar{y}, \bar{x}) + (1-\lambda) f(\bar{x}, \bar{x}) - f(\lambda \bar{y} + (1-\lambda)\bar{x}, \bar{x})] \\
&\quad - \frac{1-\lambda}{\lambda} f(\bar{x}, \bar{x}) \\
&\quad \in Y \setminus -\text{int}\, C(\bar{x}) + C(\bar{x}) + C(\bar{x}) \bigcap (-C(\bar{x})) \\
&= Y \setminus -\text{int}\, C(\bar{x})
\end{aligned}$$

contradicting (6). Thus, $\bar{x}$ is a solution of (QVI). □

REMARK 2.1. In the special case, where A is weakly compact, $f(x,y)\equiv 0$ and $K(x)\equiv A$, Theorem 2.1 collapses to Theorem 3.1 of Lin et al. (1997). Hence, Theorem 2.1 contains among others all results mentioned in Lin et al.(1997) as consequences of Theorem 3.1. Of course, in this case the coercivity condition (v) is omitted. If A is only closed and convex, a coercivity assumption is inevitable. Other observations about relations between our results and the recent ones in the literature are given in Remarks 2.2 and 2.3.

We can avoid the assumptions that $T(x)$ is compact for all x (and also weaken slightly (v) and strengthen (ii)) as follows.

THEOREM 2.2. *Assume* (iii), (iv) *and replace* (i), (ii) *and* (v) *of Theorem* 2.1, *respectively*, *by*

(i') *T is guhc in A;*
(ii') *(T,f) is pseudomonotone in A and, $\forall x\in A$, $f(.,x)$ is $C(x)$-convex in A;*
(v') *there is a nonempty weakly compact subset D of A and a subset X_0 of a weakly compact convex subset of A such that $\forall x\in A\setminus D, \exists z\in X_0\bigcap K(x), ((T(z),z-x)+f(z,x))\bigcap(-\operatorname{int}C(x))\neq\emptyset$.*

Then, (QVI) has solutions.

Proof. We use P_i,Φ_i and $Q_i, i=1,2$, as in the proof of Theorem 2.1 and define for all $x\in A$ and $y\in A$,

$$P_3(x):=\{z\in A:\exists t\in T(z):(t,z-x)+f(z,x)\in-\operatorname{int}C(x)\}.$$
$$\Phi_3(x):=\begin{cases}K(x)\bigcap P_3(x) & \text{if } x\in E,\\ A\bigcap K(x) & \text{if } x\in A\setminus E,\end{cases}$$
$$Q_3(y):=A\setminus\Phi_3^{-1}(y).$$

Then, as before Q_1 is a KKM mapping. By the pseudomonotonicity of (T,f), $A\setminus P_1^{-l}(y)\subset A\setminus P_3^{-1}(y)$ and hence $Q_1(y)\subset Q_3(y),\forall y\in A$. So, Q_3 is also a KKM mapping in A. Since Q_3 is defined also by (1), we have to show the weak closedness of $A\setminus P_3^{-1}(y),\forall y\in A$. Assume that $x_\alpha\overset{w}{\to}x\in A$ and $x_\alpha\in A\setminus P_3^{-1}(y)$, i.e. for all $t\in T(y)$ one has

$$(t,y-x_\alpha)+f(y,x_\alpha)\in Y\setminus-\operatorname{int}C(x_\alpha). \tag{7}$$

Taking (iii) into account there are a subnet x_β and $u\in-C(x)+f(y,x)$ such that $f(y,x_\beta)\overset{w}{\to}u$. By virtue of (7), of the continuity of t in the weak topologies and of the weak closedness of $Y\setminus-\operatorname{int}C(.)$ one has

$$(t,y-x)+u\in Y\setminus-\operatorname{int}C(x).$$

Therefore,

$$\begin{aligned}(t, y-x)+f(y,x) &= (t, y-x)+u+f(y,x)-u \\ &\in Y\setminus -\operatorname{int}C(x)+C(x)=Y\setminus -\operatorname{int}C(x),\end{aligned}$$

which shows that $A\backslash P_3^{-1}(y)$ is weakly closed and so is $Q_3(y)$. Similarly as for Q_2, by (v'), $\bigcap_{z\in X_0} Q_3(z)$ is weakly compact. Applying the KKM-Fan theorem gives

$$\bar{x}\in\bigcap_{y\in A}(A\setminus\Phi_3^{-1}(y))=A\setminus\bigcup_{y\in A}\Phi_3^{-1}(y).$$

Now the remaining part for indicating that $\bar{x}$ is a solution of (QVI) is similar to that of Theorem 2.1 with small modifications. □

In the following, we remove monotonicity or even its relaxed kind as assumed in Theorem 2.1 of $(T,\ f)$ by strengthening slightly the assumption on T in compensation.

THEOREM 2.3. *Impose the assumptions of Theorem* 2.1 *with the modification that the weak pseudomonotonicity in* (ii) *is deleted and the guhc of T in* (i) *is strengthened to the usc of T in the weak topology of X and norm topology of $L(X,Y)$. Then, (QVI) still has solutions.*

Proof. Observe that any $\bar{x}\in\bigcap_{y\in A} Q_1(y)$ is a solution of (QVI). To apply the KKM-Fan theorem for Q_1, it remains to check only the weak closedness of the values of Q_1. For an arbitrary $y\in A$, let $x_\alpha \overset{w}{\to} x\in A, x_\alpha\in A\backslash P_1^{-1}(y)$, i.e., there exists $t_\alpha\in T(x_\alpha)$ such that

$$(t_\alpha, y-x_\alpha)+f(y,x_\alpha)\in Y\setminus -\operatorname{int}C(x_\alpha).$$

The assumed generalized upper hemicontinuity of T implies that, $\forall\epsilon>0, \exists N(x)$ (a weak neighborhood), $T(N(x))\subset B(T(x),\epsilon)$. We can regard x_α as being in $N(x)$. Hence, there is $t'_\alpha\in T(x), \|t_\alpha - t'_\alpha\|<\epsilon$. As $T(x)$ is compact, there exist $t\in T(x)$ and subnet $t'_\beta\to t$. Consequently, $\|t_\beta - t\|\to 0$. By a similar argument as in the proof of Theorem 2.1 one sees that $x\in A\backslash P_1^{-1}(y)$. Thus, $Q_1(y)$ is closed, $\forall y\in A$. It remains to use the KKM-Fan theorem to complete the proof. □

It should be noted that the statements above are long and seemingly complicated, but the assumptions are in fact weak and not hard to be checked. We consider an example.

EXAMPLE 2.1. Let $X=Y=R$, $A=[0,1]$, $C(x)\equiv R_+$, $f(x,y)\equiv 0$,

$$K(x)=\begin{cases}\{0\} & \text{if } x\in[0,\frac{1}{100}),\\ [0,\frac{1}{5}] & \text{if } x\in(\frac{99}{100},1],\\ [0,\frac{x}{2}) & \text{otherwise,}\end{cases}$$

$$T(x)=\begin{cases}[0,x]\cup\{-1\} & \text{if } x=0 \quad\text{or } x=\frac{1}{n},\\ [0,x] & \text{otherwise.}\end{cases}$$

We check the assumptions of Theorem 2.1. (iii) and (v) are trivially satisfied. For (i), we show that T is guhc at arbitrary x_0, i.e., $\forall x\in A$, $\forall\epsilon>0, \forall\alpha>0$ small enough,

$$(T(x_0+\alpha(x-x_0)),x-x_0)\subset B((T(x_0),x-x_0),\epsilon). \tag{8}$$

We check (8) first for the case where $x_0=1/n, x\neq 1/n$. We have

$$(T(x_0+\alpha(x-x_0)),x-x_0)=\left[0,\frac{1}{n}+\alpha\left(x-\frac{1}{n}\right)\right]\left(x-\frac{1}{n}\right),$$
$$(T(x_0),x-x_0)=\left[0,\frac{1}{n}\left(x-\frac{1}{n}\right)\right]\bigcup\left\{-\left(x-\frac{1}{n}\right)\right\}.$$

So, (8) is fulfilled if

$$\frac{1}{n}\left(x-\frac{1}{n}\right)+\alpha\left(x-\frac{1}{n}\right)^2<\frac{1}{n}\left(x-\frac{1}{n}\right)+\epsilon,$$

i.e., $\alpha<\epsilon(x-1/n)^{-2}$. Next, for the case, where $x_0=0$ and $x\in A$ arbitrary, (8) holds if

$$[0,\alpha x^2]\bigcup\{-x\}\subseteq(-\epsilon,\epsilon)\bigcup(-x-\epsilon,-x+\epsilon).$$

Hence (8) is satisfied if $\alpha<\epsilon x^{-2}$. In other cases (8) is clearly fulfilled.

Assumption (ii), i.e., T is weakly pseudomonotone at any x, is obvious since we can take $t=0\in T(y)$ for all y to have $(t,y-x)\geq 0$ (see the definition).

Now for the openness of $K^{-1}(x)$ in (iv), we have three cases. If $x=0$, $K^{-1}(0)=[0,1/100)\bigcup(99/100,1]\bigcup\{y\in[0,1]:0\in[0,y/2)\}=[0,1]$. If $x\in(0,1/5]$, $K^{-1}(x)=(99/100,1]\bigcup\{y\in[0,1]:x\in[0,y/2)\}=(99/100,1)\bigcup(2y,1]$. Finally, for $x\in(1/5,1]$, $K^{-1}(x)=\{y\in[0,1]:x\in[0,y/2)\}=(2x,1]$. So $K^{-1}(x)$ is always open in A. Next for the closedness of $cl\,K(.)$ it suffices to check the usc, since A is compact. We have to consider only two points $x=1/100$

and $x = 99/100$ and the consideration is direct. The last condition in (iv) is clearly satisfied, since $E = \{0\}$. Applying Theorem 2.1, we see that (QVI) has solutions for this case. Because any solution must be in E, we conclude that the unique solution is $\bar{x} = 0$.

REMARK 2.2. (a) In Ding (1997) a scalar variational-like inequality problem, i.e. the case, where $Y = R, C(x) = R_+, K(x) \equiv K$ and $x - \bar{x}$ in the statement of our (QVI) is replaced by $\eta(x, \bar{x})$, where $\eta: K \times K \to X$ is given, is considered. His solution notion (also used in a number of other works) is stronger than that of our (QVI) and weaker than that of our (SQVI). Namely in the definition of solution, "for each $x \in K(\bar{x})$, there is $\bar{t} \in T(\bar{x})$" is replaced by "there is $\bar{t} \in T(\bar{x})$ common for all $x \in K(\bar{x})$". He does not assume weak pseudomonotonicity but needs $f(x, .)$ to be weakly continuous (this is stricter than our assumption (iii)). For a solution of our (QVI) to be a solution in the sense of Ding (1997), Lee and Kum (2000) proposed to use the Kneser minimax theorem.

(b) It is well known that equilibrium problems (EP) are more general than variational inequalities. However, the general type of variational inequalities considered in Ding (1997) and in this note contains (EP) as a special case, where $T(x) \equiv \{0\}$. So, it is interesting to compare these two ways of the problem setting. Let us mention a single example. If in our theorems $T(x) \equiv \{0\}$ and $K(x) \equiv K$, i.e. our (QVI) becomes an (EP), our results are different from Theorem 3.1, the main result of Chadli and Riahi (2000). However, our assumption (iii) is weaker than (H_1) and (H_3) of this Theorem 3.1 about semicontinuity and we do not need a pseudomonotonicity like (H_0).

(c) More general settings for (EP) in Lee and Kum (2000), Kum and Lee (2002), Fu (2000) , Fu and Wan, (2002) and Hai and Khanh (2004), including a mapping of three variables $F(t, x, y)$ instead of $f(x, y)$, reduce to our (QVI) with $K(x) \equiv K$ by taking

$$F(t, y, x) = -(t, y - x) - f(y, x). \tag{9}$$

Some results of these references are somewhat similar to ours but different. In Theorems 3.1 and 3.2, the main results of Kum and Lee (2002), $F(t, y, .)$ needs to be continuous, which is stronger than our assumptions. By taking (9), the problem setting in Hai and Khanh (in press) completely includes (QVI). However, the results are different. Our assumption that (T, f) is (weakly) pseudomonotone helps to pass the consideration to a Minty (QVI), while assumption (i) and (ii) of Theorem 3.1 of Hai and Khanh (in press) are on another kind of pseudomonotonicity. In our (i) and (iii), T is generalized usc in directions and f satisfies a condition weaker than lsc. The corresponding assumption (iii) of Hai and Khanh (in press) is a kind of lsc (for T and f together), and then is stronger. Our assumptions (v) or (v') is a coercivity condition for the mentioned Minty (QVI), not a "direct coercivity" as in Hai and Khanh (in press). However, we can observe similar techniques in Hai and Khanh (in press) and this note. They are similar to that of a number of works in the literature, using the well-known KKM-Fan theorem.

3. Existence for (SQVI)

THEOREM 3.1. *Let all the assumptions of Theorem* 2.2 *be satisfied with*

(i') *replaced by*
(i") *T is glhc in A.*

Then, Problem (SQVI) has solutions.

Proof. As in the proof of Theorem 2.2, one obtains a point

$$\bar{x} \in \bigcap_{y \in A} (A \setminus \Phi_3^{-1}(y)) = A \setminus \bigcup_{y \in A} \Phi_3^{-1}(y).$$

Hence, $\Phi_3(\bar{x}) = \emptyset$ and as before $\bar{x}$ must be in E. Then, $\emptyset = \Phi_3(\bar{x}) = K(\bar{x}) \bigcap P_2(\bar{x})$. So, for each $y \in K(\bar{x}), y \notin P_3(\bar{x})$, i.e.,

$$(T(y), y - \bar{x}) + f(y, \bar{x}) \subset Y \setminus -\operatorname{int} C(\bar{x}). \tag{10}$$

For each $y \in K(\bar{x})$, define $G(\lambda) := (T(\lambda y + (I - \lambda)\bar{x}), y - \bar{x})$. By (i"), G is lsc at 0^+, i.e., $\forall g \in G(0), \forall \lambda_n \to 0^+, \exists g_n \in G(\lambda_n), g_n \to g$. Setting $y_n := \lambda_n y + (1 - \lambda_n)\bar{x}$, one has $t_n \in T(y_n)$ such that $g_n = (t_n, y - \bar{x})$. By (iv), $y_n \in K(\bar{x})$. Then, (10) implies that

$$(t_n, y_n - \bar{x}) + f(y_n, \bar{x}) \in Y \setminus -\operatorname{int} C(\bar{x}).$$

Therefore,

$$\begin{aligned}
&g_n + f(y, \bar{x}) \\
&\quad = \frac{1}{\lambda_n}[(t_n, \lambda_n y + (1 - \lambda_n)\bar{x} - \bar{x}) + f(y, \bar{x})] \\
&\quad = \frac{1}{\lambda_n}[(t_n, y_n - \bar{x}) + f(y_n, \bar{x})] + \frac{1}{\lambda_n}[\lambda_n f(y, \bar{x}) + (1 - \lambda_n) f(\bar{x}, \bar{x}) \\
&\quad - f(\lambda_n y + (1 - \lambda_n)\bar{x}, \bar{x})] - \frac{1 - \lambda_n}{\lambda_n} f(\bar{x}, \bar{x}) \\
&\quad \in Y \setminus -\operatorname{int} C(\bar{x}) + C(\bar{x}) + C(\bar{x}) \bigcap (-C(\bar{x})) \\
&\quad = Y \setminus -\operatorname{int} C(\bar{x}).
\end{aligned}$$

Letting $n \to \infty$, one obtains $g + f(y, \bar{x}) \in Y \setminus -\operatorname{int} C(\bar{x})$. This is true for all $y \in K(\bar{x})$ and all $g \in G(0) = (T(\bar{x}), y - \bar{x})$, i.e., $\bar{x}$ is a solution of (SQVI). □

4. Applications to Traffic Network Problems

Wardrop (1952) introduced a notion of equilibrium flow for transportation network problems. Based on this definition these trafic problems with a single criterion for equilibrium, i.e. with a scalar cost, have been studied extensively in both theory and methodology view points (see discussions and surveys e.g. in Florian, 1986; Nagurney, 1993). Since many criteria (such as time delay, required toll…) should be compromised in practical problems, Chen and Yen (1993) proposed an extension of the Wardrop notion of equilibrium to the vector case. See Yang and Goh (1997) and Goh and Yang (1999) for remarkable developments in this direction. In particular, the papers make it clear that a scalar traffic network problem is equivalent to variational inequality problem, but this equivalence is no longer true for the two corresponding vector problems. Relations among the solutions of these two problems and that of the corresponding vector optimization problems are also discussed in some details. In Daniele et al., (1999), the scalar traffic model is extended to the case, where the data (i.e. the flow capacity, and the demand) are time-dependent. It is proved that the mentioned equivalence holds for this case. See also Giannessi (2000) for various developments of traffic network considerations.

De Luca (1995) and Maugeri (1995) proposed to consider the costs as multifunctions of the flows and the demands depending on the equilibrium flow to make the traffic model more elastic and suitable for diverse practical situations. Developing this idea, Khanh and Luu (2004) extended the Wardrop definition of equilibrium to weak equilibrium and strong equilibrium, suitable for this model of multivalued costs. Sufficient conditions for the existence of weak and strong equilibria are established based on the observation that the weak and strong solutions of the multivalued vector quasivariational inequality problem are, respectively, weak and strong equilibrium flows.

In this section, we illustrate applications of the results obtained in the previous sections in two ways. First we show that applying Theorems 2.1 and 3.1 to the traffic problem considered in Khanh and Luu (2004) we obtain again the results of that paper. Next we show that due to the fact that the quasivariational inequality problem in this paper is more general than that in Khanh and Luu (2004) we can apply it to consider in more details the traffic problem with a particular tolerance in satisfying the demands. Since our goal here is not related directly to whether the problem is scalar or vector, for the sake of simplicity we consider the scalar case.

Recall first the traffic network problem considered in Khanh and Luu (2004). Let N be the set of nodes, L be that of links (or arcs), $W := (W_1, \ldots, W_l)$ be the set of origin-destination pairs (O/D pairs for short). Assume that the pair W_j, $j = 1, \ldots, l$, is connected by a set P_j of paths (P_j containing $r_j \geq 1$ paths). Set $m := r_1 + \cdots + r_l$. Let $F := (F_1, \ldots, F_m)$ be the path flow vector. Let the cost vector $T(F) := (T_1(F), \ldots, T_m(F))$ be given as a multifunction $T: R^m_+ \overrightarrow{\rightarrow} R^m_+$. Let Γ_s be the capacity of path R_s, $s = 1, \ldots, m$. Then the flows must satisfy the constraint

$$F \in A := \{F \in R^m_+ : 0 \leqslant F_s \leqslant \Gamma_s, s = 1, \ldots, m\}. \tag{11}$$

We propose the following generalization of the Wardrop equilibrium.

DEFINITION 4.1. (i) *A path flow vector H is said to be a weak equilibrium flow vector if* $\forall W_j, \forall R_q \in P_j, \forall R_s \in P_j, \exists t \in T(H)$,

$$t_q < t_s \Rightarrow H_q = \Gamma_q \qquad \text{or} \qquad H_s = 0,$$

where $j = 1, \ldots, l$ and $q, s \in \{1, \ldots, m\}$ are among r_j indeces corresponding to P_j.

(ii) *A path flow vector H is called a strong equilibrium flow vector if* (i) *is satisfied with $\exists t \in T(H)$ being replaced by $\forall t \in T(H)$.*

Following De Luca (1995) and Maugeri (1995), the demands ρ_j of the O/D pairs W_j may depend on the equilibrium flow vector H. So we have a mapping $\rho: R^m_+ \to R^l_+$, which is assumed to be continuous. Set

$$\phi_{js} := \begin{cases} 1 & \text{if } R_s \in P_j, \\ 0 & \text{if } R_s \notin P_j, \end{cases}$$
$$\phi = \{\varphi_{js}\}, \quad j = 1, \ldots, l, \quad s = 1, \ldots, m.$$

Let $\epsilon: R^m_+ \to R_+$ be a continuous functional. Assume that there is a tolerance in demands such that the set of the feasible path flow vectors is

$$K(H) := \{F \in R^m_+ : \phi F \in B(\rho(H), \epsilon(H)), F \in A\},$$

where $B(\rho(H), \epsilon(H))$ is the ball (in R^l) centered at $\rho(H)$ with radius $\epsilon(H)$. Then $K(.)$ satisfied (iv) of Theorem 2.1 as shown in Khanh and Luu (2004).

Observe that a feasible path flow vector $\bar{H}$ is a weak equilibrium flow vector if $\bar{H}$ is a solution of (QVI) problem: finding $\bar{H} \in K(\bar{H})$ such that for each $F \in K(\bar{H})$, there is $\bar{t} \in T(\bar{H})$ such that

$$< \bar{t}, F - \bar{H} > \geqslant 0.$$

Repeating almost the same arguments as in Khanh and Luu (2004), instead of applying Theorems 2.1 and 2.2 there, we can use Theorems 2.1 and 3.1 of this paper to get the following existence results.

COROLLARY 4.1. *If T is guhc and pseudomonotone in A and has compact values, then the traffic network has a weak equilibrium flow vector.*

COROLLARY 4.2. *If T is glhc and pseudomonotone in A, then the traffic network has a strong equibrium flow vector.*

REMARK 4.1. (i) The advantage of Theorems 2.1 and 3.1 here is that they guarantee that $E \neq \emptyset$, while applying the results of Khanh and Luu (2004) one has to assume that $\Gamma_s, \rho(.)$ and $\epsilon(.)$ are given so that $E \neq \emptyset$, which is not verifiable.

(ii) As mentioned at the beginning of this section, the classical scalar traffic network problem (with single-valued costs and fixed demands) is equivalent to the corresponding variational inequality problem. However, for the above network problem we have only one implication: any weak or strong solution of (QVI) is a weak or strong, respectively, equilibrium flow vector. Now we establish the reverse implication for the following special case.

Assume that the ball $B(\rho, \epsilon)$ in R^l (of the tolerance in the demands) has the form $\{v \in R^l : \max_{i=1,\dots,l} |v_i - \rho_i| < \epsilon\}$. Assume also that

$$M := \sup_{H \in A} \sup_{t \in T(H)} \sup_{i=1,\dots,m} |t_p| < +\infty.$$

PROPOSITION 4.3. *If $\bar{H}$ is a weak equilibrium flow vector, then $\bar{H}$ is a solution of the quasivariational inequality problem (QVI) with $f(x, y) = 2Ml\epsilon(y)$. In fact $\bar{H}$ is a solution of the more stronger problem of finding $\bar{H} \in A \cap clK(\bar{H})$ such that there is $\bar{t} \in T(\bar{H})$ satisfying, for all $F \in K(\bar{H})$,*

$$\langle \bar{t}, F - \bar{H} \rangle + 2Ml\epsilon(\bar{H}) \geq 0. \tag{12}$$

Proof. Assume that $\bar{H}$ is a weak equilibrium flow. For each O/D pair W_j set

$$A_j := \{R_q \in P_j : \bar{H}_q < \Gamma_q\},$$
$$B_j : = \{R_s \in P_j : \bar{H}_s > 0\}.$$

Then, by the definition of weak equilibrium flows, $\bar{t}_q \geqslant \bar{t}_s$ if $R_q \in A_j$ and $R_s \in B_j$. Take γ_j such that

$$\inf_{R_q \in A_j} \{\bar{t}_q\} \geqslant \gamma_j \geqslant \sup_{R_s \in B_j} \{\bar{t}_s\}.$$

We have to check (12) for arbitrary $F \in K(\bar{H})$. Let a pair W_j and a path $R_r \in W_j$ be arbitrary. If $\bar{t}_r < \gamma_j$, then $R_r \notin A_j$, i.e. $\bar{H}_r = \Gamma_r$, and hence

$$(\bar{t}_r - \gamma_j)(F_r - \bar{H}_r) \geq 0. \tag{13}$$

If $\bar{t}_r > \gamma_j$, then $R_r \notin B_j$, i.e. $\bar{H}_r = 0$, and hence (13) holds. If $\bar{t}_r = \gamma_j$, (13) clearly holds. Since (13) is satisfied for all W_j and all $R_r \in W_j$ one has

$$\begin{aligned}
&\langle \bar{t}, F - \bar{H} \rangle \\
&= \sum_{j=1}^{l} \sum_{R_r \in W_j} \bar{t}_r (F_r - \bar{H}_r) \\
&\geq \sum_{j=1}^{l} \gamma_j \sum_{R_r \in W_j} (F_r - \bar{H}_r) \\
&\geq \sum_{j=1}^{l} \gamma_j (\rho_j(\bar{H}) - \epsilon(\bar{H}) - (\rho_j(\bar{H}) + \epsilon(\bar{H}))) \\
&= -2\epsilon(\bar{H}) \sum_{j=1}^{l} \gamma_j .
\end{aligned}$$

Therefore,

$$\langle \bar{t}, F - \bar{H} \rangle + 2Ml\epsilon(\bar{H}) \geq 2\epsilon(\bar{H}) \left(Ml - \sum_{j=1}^{l} \gamma_j \right) \geq 0. \qquad \square$$

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